MTH 295
Fall 2019
Homework 3
Due Thursday, 9/26 Name:___________________________

1) a) Show that the ODE $ydx - xdy = 0$ is not exact.

This equation is of the form $Mdx + Ndy = 0$ where $M = y$, $N = -x$

$\frac{\partial}{\partial y}(y) = 1$

$\frac{\partial}{\partial x}(-x) = -1$

So the equation is not exact.

b) Fortunately it need not be exact for you to solve it. Find the general solution of the above equation.

This equation is separable -

$ydx - xdy = 0$

$ydx = xdy$

$\frac{dy}{y} = \frac{dx}{x}$

$\ln|y| = \ln|x| + c$

$|y| = c|x|$

or $\boxed{y = cx}$

aside —
notice there are 4 cases -

if $x > 0, y > 0$ then
$|y| = c|x|$
$y = cx$

if $x < 0, y < 0$ then
$|y| = c|x|$
$-y = c(-x)$
$y = cx$

if $x > 0, y < 0$
$|y| = c|x|$
$-y = cx$
$y = -cx$
$= cx$ because $-c = c$
$=$ constant

if $x < 0, y > 0$
$|y| = c|x|$
$y = c(-x)$
$= -cx$
$= cx$ as above

2) Show that the equation $(x+\sin y)dx+(x\cos y-2y)dy=0$ is exact and find its general solution. Your solution will be implicit.

$\frac{\partial}{\partial y}(x+\sin y)=\cos y$

$\frac{\partial}{\partial x}(x\cos y-2y)=\cos y$

So there exists an $F(x,y)=c$ such that

$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=(x+\sin y)dx+(x\cos y-2y)dy=0$

so $\frac{\partial F}{\partial x}=x+\sin y$

$F=\frac{x^2}{2}+x\sin y+g(y)$

$\frac{\partial F}{\partial y}=x\cos y+g'(y)=x\cos y-2y$

$g'(y)=-2y$

$g(y)=-y^2$

so $F(x,y)=\frac{x^2}{2}+x\sin y-y^2$

and $\boxed{\frac{x^2}{2}+x\sin y-y^2=c}$

3) a) Show that $x^2ydy - xy^2dx - x^3y^2dx = 0$ is not exact.

$(x^3y^2 + xy^2)dx - x^2ydy = 0$

$\frac{\partial}{\partial y}(x^3y^2 + xy^2) = 2x^3y + 2xy$

but $\frac{\partial}{\partial x}(-x^2y) = -2xy$

so the equation is not exact

b) Show that $\mu(x,y) = \dfrac{1}{x^2y^2}$ is an integrating factor that makes the equation exact.

multiplying by $\mu(x,y)$ gives

$(x + \frac{1}{x})dx - \frac{1}{y}dy = 0$

$\frac{\partial}{\partial y}(x + \frac{1}{x}) = \frac{\partial}{\partial x}(-\frac{1}{y}) = 0$ so the new equation is exact.

c) Find the general solution. Express y as an explicit function of x.

Since $(x + \frac{1}{x})dx - \frac{1}{y}dy = 0$ is exact there exists a function $F(x,y) = C$ such that

$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = (x + \frac{1}{x})dx - \frac{1}{y}dy = 0$

so $\frac{\partial F}{\partial x} = x + \frac{1}{x}$

$F = \frac{x^2}{2} + \ln|x| + g(y)$

$\frac{\partial F}{\partial y} = g'(y) = -\frac{1}{y}$

$g(y) = -\ln|y|$

$F = \frac{x^2}{2} + \ln|x| - \ln|y| = C$

$\left|\frac{x}{y}\right| = Ce^{-x^2/2}$

$\frac{x}{y} = Ce^{-x^2/2}$ (see aside on problem 1)

and $\boxed{y = Cxe^{x^2/2}}$

4) Solve the IVP $y' + \frac{2}{x}y = x$, $y(1) = 0$.

This equatión is linear (you check it's not exact)

So there exists an integrating factor

$$\mu(x) = e^{\int \frac{2}{x}dx}$$

$$= e^{2\ln|x|}$$

$$= e^{\ln x^2}$$

$$= x^2$$

such that the equatión is exact and

$$\frac{d}{dx}(\mu y) = \mu x$$

$$\frac{d}{dx}(x^2 y) = x^3$$

$$x^2 y = \frac{x^4}{4} + C$$

apply the IC –

$$0 = \frac{1}{4} + C$$

$$C = -\frac{1}{4}$$

$$x^2 y = \frac{x^4}{4} - \frac{1}{4}$$

$$\boxed{y = \frac{x^2}{4} - \frac{1}{4x^2}}$$

5) Solve the IVP $\frac{dx}{dt} - (\tan t)x = \sin t$, $x(0) = 1$.

This equation is linear so there exists an integrating factor

$$\mu(x) = e^{-\int \tan t\, dt}$$

$$= e^{-\ln|\sec t|}$$

$$= e^{\ln|\cos t|}$$

$$= |\cos t|$$

you may take $\mu = \cos t$ because it is also an integrating factor (you verify this!)

so $\frac{d}{dt}(x \cos t) = \sin t \cdot \cos t$

$$x \cos t = \frac{\sin^2 t}{2} + c$$

apply the condition $x(0) = 1$

$$(1)\cos(0) = \frac{\sin^2(0)}{2} + c$$

$$c = 1$$

$$x \cos t = \frac{\sin^2 t}{2} + 1$$

$$\boxed{x = \frac{1}{2} \sin t \tan t + \sec t}$$